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Regularity of invariant sets in semilinear damped wave equations

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ABSTRACT

Under fairly general assumptions, we prove that every compact invariant subset $\mathcal{I}$ of the semiflow generated by the semilinear damped wave equation

$$\epsilon u_{tt} + u_t + \beta(x)u - \sum_{ij} \big(a_{ij}(x)u_{x_j}\big)_{x_i} = f(x,u),$$
$$(t,x) \in [0,+\infty[\times \Omega,$$
$$u = 0, \quad (t,x) \in [0,+\infty[\times \partial\Omega,$$

in $H_0^1(\Omega) \times L^2(\Omega)$ is in fact bounded in $D(\mathbf{A}) \times H_0^1(\Omega)$. Here Ω is an arbitrary, possibly unbounded, domain in $\mathbb{R}^3$, $\mathbf{A}u = \beta(x)u - \sum_{ij}(a_{ij}(x)u_{x_j})_{x_i}$ is a positive selfadjoint elliptic operator and $f(x,u)$ is a nonlinearity of critical growth. The nonlinearity $f(x,u)$ needs not to satisfy any dissipativeness assumption and the invariant subset $\mathcal{I}$ needs not to be an attractor.

1. Introduction

Consider the semilinear damped wave equation

$$\begin{aligned} &\epsilon u_{tt} + u_t + \beta(x)u - \sum_{ij} \big(a_{ij}(x)u_{x_j}\big)_{x_i} = f(x,u), \quad (t,x) \in [0,+\infty[\times \Omega, \\ &u = 0, \quad (t,x) \in [0,+\infty[\times \partial\Omega, \end{aligned} \tag{1.1}$$

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where Ω is an arbitrary, possibly unbounded, domain in $\mathbb{R}^3$, $f(x,u)$ is a nonlinearity of critical growth and $\mathbf{A}u := \beta(x)u - \sum_{ij}(a_{ij}(x)u_{x_j})_{x_i}$ is a positive selfadjoint elliptic operator. It is well known (see e.g. [18]) that Eq. (1.1), under appropriate conditions on $a_{ij}(x)$, $\beta(x)$ and $f(x,u)$, generates a (local) semiflow in the space $H_0^1(\Omega) \times L^2(\Omega)$. We remind that a subset $\mathcal{S}$ of $H_0^1(\Omega) \times L^2(\Omega)$ is called *invariant* for the semiflow generated by (1.1) if for every $(u_0, v_0) \in \mathcal{S}$ there exists a solution $(u(\cdot), v(\cdot)) : \mathbb{R} \to H_0^1(\Omega) \times L^2(\Omega)$ of (1.1) with $(u(0), v(0)) = (u_0, v_0)$ and $(u(t), v(t)) \in \mathcal{S}$ for all $t \in \mathbb{R}$. Assume that $\mathcal{I}$ is a compact invariant subset for this semiflow. In this paper we shall prove that, under fairly general assumptions on $a_{ij}(x)$, $\beta(x)$ and $f(x,u)$, $\mathcal{I}$ is in fact bounded in $D(\mathbf{A}) \times H_0^1(\Omega)$. This means that a solution of (1.1) lying in $\mathcal{I}$ is more regular in space than a generic solution. Results of this kind have been known for a long time in case $f(x,u)$ satisfies some dissipativeness condition and $\mathcal{I}$ is the global attractor of (1.1) (see e.g. [1,4,5,9,11] and the more recent [3,7,8,13,19]). In [17] regularity results were obtained for general invariant subsets in the subcritical case. To our knowledge, the most general results are contained in the paper [10] by Hale and Raugel, where the authors, among other things, prove "spatial regularity" of invariant subsets for a general class of abstract semilinear evolution equations. The equations considered by Hale and Raugel are of the form $\dot{u} = Au + f(u)$, where A is the generator of a C^0-semigroup of linear operators in a Banach space X and f is a nonlinearity of class $C^{1,1}$. The assumptions in [10] are too elaborated to be summarized here. The technique relies on suitable Galerkin decompositions of the solutions lying in the invariant subset. Roughly speaking, every solution $u(t)$ in the invariant subset splits as $u(t) = v(t) + w(t)$, where w is the fixed point of an integral equation and $v(t)$ is the solution of a retarded differential equation on a (usually finite-dimensional) subspace of X. The applications described in [10] consider only the case of equations on bounded domains, where a natural Galerkin decomposition is supplied by the (finite-dimensional) spectral projections. However, it is very likely that the abstract results of [10] should apply also to the case of equations on unbounded domains. In this case, the decomposition on a basis of eigenfunctions should be replaced by the use of the spectral family of the operator A.

Our aim is to go beyond the results of [10] in the particular case of the semilinear damped wave equation (1.1). We shall prove our regularity results without any smoothness and/or boundedness assumption on Ω. The nonlinearity $f(x,u)$ needs not to be of class $C^{1,1}$ in u, but only of class $C^{1,\beta}$ for some $0 < \beta < 1$. Moreover, we shall not exploit Galerkin decompositions of the solutions, so we bypass the problem of constructing spectral families. Finally, we do not need to use the theory of retarded differential equations.

The idea of the proof is very simple, although it requires a careful functional analytic setting. We give here an informal sketch. Let $(\bar{u}(\cdot), \bar{u}_t(\cdot)) : \mathbb{R} \to H_0^1(\Omega) \times L^2(\Omega)$ be a bounded mild solution of (1.1). Set $\bar{v}(t) := \bar{u}_t(t)$. Then $(\bar{v}(\cdot), \bar{v}_t(\cdot)) : \mathbb{R} \to L^2(\Omega) \times H^{-1}(\Omega)$ is a mild solution of

$$\begin{aligned} \epsilon v_{tt} + v_t + \beta(x)v - \sum_{ij}\big(a_{ij}(x)v_{x_j}\big)_{x_i} &= \partial_u f\big(x, \bar{u}(t)\big)v, \quad (t,x) \in [0, +\infty[\times \Omega, \\ v &= 0, \quad (t,x) \in [0, +\infty[\times \partial\Omega. \end{aligned} \tag{1.2}$$

Take $\theta > 0$ and denote by $\mathbf{U}(t,s)$ the evolution system generated by the non-autonomous linear equation

$$\begin{aligned} \epsilon v_{tt} + v_t + \beta(x)v - \sum_{ij}\big(a_{ij}(x)v_{x_j}\big)_{x_i} + \theta v - \partial_u f\big(x, \bar{u}(t)\big)v &= 0, \quad (t,x) \in [0, +\infty[\times \Omega, \\ v &= 0, \quad (t,x) \in [0, +\infty[\times \partial\Omega, \end{aligned} \tag{1.3}$$

in the space $L^2(\Omega) \times H^{-1}(\Omega)$. Then, for $t \geqslant s$, we have that

$$\big(\bar{v}(t), \bar{v}_t(t)\big) = \mathbf{U}(t,s)\big(\bar{v}(s), \bar{v}_t(s)\big) + \int_s^t \mathbf{U}(t,p)\big(0, (\theta/\epsilon)\bar{v}(p)\big)\,dp. \tag{1.4}$$

We shall prove in Proposition 3.7 below that, if θ is sufficiently large, then $\mathbf{U}(t,s)$ satisfies appropriate exponential decay estimates in $L^2(\Omega)\times H^{-1}(\Omega)$ as well as in $H_0^1(\Omega)\times L^2(\Omega)$. Then, letting $s\to-\infty$ in (1.4), we obtain that

$$\bigl(\bar v(t),\bar v_t(t)\bigr)=\int_{-\infty}^{t}\mathbf{U}(t,p)\bigl(0,(\theta/\epsilon)\bar v(p)\bigr)\,dp. \tag{1.5}$$

In this way we get rid of the Cauchy data $(\bar v(s),\bar v_t(s))$ and, since $(0,(\theta/\epsilon)\bar v(p))\in H_0^1(\Omega)\times L^2(\Omega)$, we deduce that actually $(\bar v(\cdot),\bar v_t(\cdot))$ is a bounded function from $\mathbb{R}$ into $H_0^1(\Omega)\times L^2(\Omega)$ and the conclusion follows. It is likely that one could adapt this argument to other problems (e.g. wave equations with localized or boundary damping). A similar idea was already exploited in [4].

The paper is organized as follows. In Section 2 we introduce notations, we set the main assumptions and we state the main results. In Section 3 we study in detail the evolution system generated by the non-autonomous linear equation (1.3). At a first stage, the reader can skip the proofs or the results contained in Section 3 and proceed directly to Sections 4 and 5, which are devoted to the proofs of the main results. Finally, in Section 6 we exploit the regularity results to prove upper semicontinuity of the attractors of (1.1) as $\epsilon\to 0$ when $f(x,u)$ is dissipative, thus improving a previous result obtained with K. Rybakowski [16].

2. Notation, statements and remarks

Before we describe in detail our assumptions and our results, we need to introduce some notation. In this paper Ω is an arbitrary open subset of $\mathbb{R}^3$, bounded or not. Given a function $g:\Omega\times\mathbb{R}\to\mathbb{R}$, we denote by $\hat g$ the Nemitski operator which associates with every function $u:\Omega\to\mathbb{R}$ the function $\hat g(u):\Omega\to\mathbb{R}$ defined by

$$\hat g(u)(x)=g\bigl(x,u(x)\bigr),\quad x\in\Omega.$$

If $I\subset\mathbb{R}$, Y and X are normed spaces with $Y\subset X$ and if $u:I\to Y$ is a function which is differentiable as a function into X then we denote its X-valued derivative by $(\partial_t\mid X)u$. Similarly, if X is a Banach space and $u:I\to X$ is integrable as a function into X, then we denote its X-valued integral by $\int_I u(t)(dt\mid X)$. If X and Y are Banach spaces, we denote by $\mathcal{L}(X,Y)$ the space of bounded linear operators from X to Y. If $X=Y$ we write just $\mathcal{L}(X)$.

Hypothesis 2.1.

(1) $a_0,a_1\in\,]0,\infty[$ are constants and $a_{ij}:\Omega\to\mathbb{R}$ are functions in $L^\infty(\Omega)$ such that $a_{ij}=a_{ji}$, $i,j=1,\dots,3$, and for every $\xi\in\mathbb{R}^3$ and a.e. $x\in\Omega$,

$$a_0|\xi|^2\leqslant\sum_{i,j=1}^{3}a_{ij}(x)\xi_i\xi_j\leqslant a_1|\xi|^2.$$

(2) $\beta:\Omega\to\mathbb{R}$ is a measurable function with the property that
 (a) for every $\nu>0$ there is a $C_\nu>0$ with

$$\int_\Omega\bigl|\beta(x)\bigr|\bigl|u(x)\bigr|^2\,dx\leqslant\nu\int_\Omega\bigl|\nabla u(x)\bigr|^2\,dx+C_\nu\int_\Omega\bigl|u(x)\bigr|^2\,dx$$

 for all $u\in H_0^1(\Omega)$;

(b) there exists $\lambda_1 > 0$ such that, setting $A(x) := (a_{ij}(x))_{i,j=1}^3$,

$$\int_\Omega A(x)\nabla u(x) \cdot \nabla u(x)\,dx + \int_\Omega \beta(x)\left|u(x)\right|^2 dx \geqslant \lambda_1 \int_\Omega \left|u(x)\right|^2 dx$$

for all $u \in H_0^1(\Omega)$.

Remark 2.2. Condition (a) in Hypothesis 2.1 is satisfied, e.g., if $\beta \in L_{\mathrm{u}}^p(\mathbb{R}^3)$ with $p > 3/2$. Here we denote by $L_{\mathrm{u}}^p(\mathbb{R}^3)$ the set of measurable functions $\zeta : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\|\zeta\|_{L_{\mathrm{u}}^p} := \sup_{y \in \mathbb{R}^3} \left(\int_{B(y)} \left|\zeta(x)\right|^p dx \right)^{1/p} < \infty,$$

where, for $y \in \mathbb{R}^3$, $B(y)$ is the open unit cube in $\mathbb{R}^3$ centered at y (see [15] for details).

By Lemma 3.4 in [15], the scalar product

$$\langle u, v\rangle_{H_0^1} = \int_\Omega A(x)\nabla u(x) \cdot \nabla v(x)\,dx + \int_\Omega \beta(x)u(x)v(x)\,dx, \quad u, v \in H^1(\Omega), \tag{2.1}$$

is equivalent to the usual scalar product on $H_0^1(\Omega)$. From now on, we denote by $\|\cdot\|_{H_0^1}$ the norm associated with $\langle\cdot,\cdot\rangle_{H_0^1}$.

Let $\mathbf{A}$ be the selfadjoint operator on $L^2(\Omega)$ defined by the differential operator $u \mapsto \beta u - \sum_{ij}(a_{ij}u_{x_j})_{x_i}$. Then $\mathbf{A}$ generates a family X^κ, $\kappa \in \mathbb{R}$, of fractional power spaces with $X^{-\kappa}$ being the dual of X^κ for $\kappa \in]0, +\infty[$. For $\kappa \in]0, +\infty[$, the space X^κ is a Hilbert space with respect to the scalar product

$$\langle u, v\rangle_{X^\kappa} := \left\langle \mathbf{A}^\kappa u, \mathbf{A}^\kappa v\right\rangle_{L^2}, \quad u, v \in X^\kappa.$$

Also, the space $X^{-\kappa}$ is a Hilbert space with respect to the scalar product $\langle\cdot,\cdot\rangle_{X^{-\kappa}}$ dual to the scalar product $\langle\cdot,\cdot\rangle_{X^\kappa}$, i.e.

$$\langle u', v'\rangle_{X^{-\kappa}} = \left\langle R_\kappa^{-1}u', R_\kappa^{-1}v'\right\rangle_{X^\kappa}, \quad u, v \in X^{-\kappa},$$

where $R_\kappa : X^\kappa \to X^{-\kappa}$ is the Riesz isomorphism $u \mapsto \langle\cdot, u\rangle_{X^\kappa}$.

We write

$$H_\kappa = X^{\kappa/2}, \quad \kappa \in \mathbb{R}.$$

Note that $H_0 = L^2(\Omega)$, $H_1 = H_0^1(\Omega)$, $H_{-1} = H^{-1}(\Omega)$ and $H_2 = D(\mathbf{A})$.

For $\kappa \in \mathbb{R}$ the operator $\mathbf{A}$ induces a selfadjoint operator $\mathbf{A}_\kappa : H_{\kappa+2} \to H_\kappa$. In particular $\mathbf{A} = \mathbf{A}_0$. Moreover,

$$\langle u, v\rangle_{H_0^1} = \langle \mathbf{A}_0 u, v\rangle_{L^2}, \quad u \in D(\mathbf{A}_0),\ v \in H_0^1(\Omega).$$

For $\epsilon \in]0, 1]$ and $\kappa \in \mathbb{R}$ set $Z_\kappa := H_{\kappa+1} \times H_\kappa$ and define the linear operator $\mathbf{B}_{\epsilon,\kappa} : Z_{\kappa+1} \to Z_\kappa$ by

$$\mathbf{B}_{\epsilon,\kappa}(u,v) := \big(v, -(1/\epsilon)(v + \mathbf{A}_\kappa u)\big), \quad (u,v) \in Z_{\kappa+1}.$$

It follows that $\mathbf{B}_{\epsilon,\kappa}$ is m-dissipative on Z_κ (cf. the proof of Proposition 3.6 in [15]). Therefore, by the Hille–Yosida–Phillips theorem (see e.g. [2]), $\mathbf{B}_{\epsilon,\kappa}$ is the infinitesimal generator of a C^0-semigroup $\mathbf{T}_{\epsilon,\kappa}(t)$, $t \in [0,+\infty[$, on Z_κ.

Hypothesis 2.3.

(1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is such that, for every $u \in \mathbb{R}$, $f(\cdot,u)$ is measurable and $f(\cdot,0) \in L^2(\Omega)$;
(2) for a.e. $x \in \Omega$, $f(x,\cdot)$ is of class C^1, $\partial_u f(\cdot,0) \in L^\infty(\Omega)$ and there exist constants C, β and α, with $C > 0$, $0 < \beta \leqslant 1$, $1 \leqslant \alpha < 2$ and $\alpha + \beta = 2$, such that

$$\big|\partial_u f(x,u_1) - \partial_u f(x,u_2)\big| \leqslant C\big(1 + |u_1|^\alpha + |u_2|^\alpha\big)|u_1 - u_2|^\beta.$$

The main properties of the Nemitski operator associated with f are collected in the following proposition, whose proof is left to the reader.

Proposition 2.4. *Assume Hypothesis* 2.3. *Then* $\hat{f} : H_0^1(\Omega) \to L^2(\Omega)$ *is continuously differentiable,* $D\hat{f}(u)[v](x) = \partial_u f(x,u(x))v(x)$ *for* $u, v \in H_0^1(\Omega)$, *and there exists a positive constant* $\tilde{C} > 0$ *such that the following estimates hold*:

$$\big\|\hat{f}(u)\big\|_{L^2} \leqslant \tilde{C}\big(1 + \|u\|^3_{H_0^1}\big), \quad u \in H_0^1(\Omega), \tag{2.2}$$

$$\big\|D\hat{f}(u)\big\|_{\mathcal{L}(H_0^1,L^2)} \leqslant \tilde{C}\big(1 + \|u\|^2_{H_0^1}\big), \quad u \in H_0^1(\Omega), \tag{2.3}$$

$$\big\|D\hat{f}(u_1) - D\hat{f}(u_2)\big\|_{\mathcal{L}(H_0^1,L^2)} \leqslant \tilde{C}\big(1 + \|u_1\|^\alpha_{H_0^1} + \|u_\alpha\|^\alpha_{H_0^1}\big)\|u_1 - u_2\|^\beta_{H_0^1}, \quad u_1, u_2 \in H_0^1(\Omega). \tag{2.4}$$

If $u \in H_0^1(\Omega)$ *and* $v \in L^2(\Omega)$, *then* $\widehat{\partial_u f}(u) \cdot v \in H^{-1}(\Omega)$ *and the following estimates hold*:

$$\big\|\widehat{\partial_u f}(u)\big\|_{\mathcal{L}(L^2,H^{-1})} \leqslant \tilde{C}\big(1 + \|u\|^2_{H_0^1}\big), \quad u \in H_0^1(\Omega), \tag{2.5}$$

$$\big\|\widehat{\partial_u f}(u_1) - \widehat{\partial_u f}(u_2)\big\|_{\mathcal{L}(L^2,H^{-1})} \leqslant \tilde{C}\big(1 + \|u_1\|^\alpha_{H_0^1} + \|u_2\|^\alpha_{H_0^1}\big)\|u_1 - u_2\|^\beta_{H_0^1}, \quad u_1, u_2 \in H_0^1(\Omega). \tag{2.6}$$

Finally, whenever the function $t \mapsto u(t)$ *is continuous from* $\mathbb{R}$ *to* $H_0^1(\Omega)$ *and continuously differentiable from* $\mathbb{R}$ *to* $L^2(\Omega)$, *then the function* $t \mapsto \hat{f}(u(t))$ *is continuously differentiable from* $\mathbb{R}$ *to* $H^{-1}(\Omega)$ *and*

$$\big(\partial_t \mid H^{-1}\big)(\hat{f} \circ u)(t) = \widehat{\partial_u f}\big(u(t)\big) \cdot \big(\partial_t \mid L^2\big)u(t). \tag{2.7}$$

We consider the following semilinear damped wave equation:

$$\begin{gathered}\epsilon u_{tt} + u_t + \beta(x)u - \sum_{ij}\big(a_{ij}(x)u_{x_j}\big)_{x_i} = f(x,u), \quad (t,x) \in [0,+\infty[\times \Omega, \\ u = 0, \quad (t,x) \in [0,+\infty[\times \partial\Omega,\end{gathered} \tag{2.8}$$

with Cauchy data $u(0) = u_0$, $u_t(0) = v_0$.

Remark 2.5. The condition $\lambda_1 > 0$ in Hypothesis 2.1 is not restrictive. Indeed, if Hypothesis 2.1 is satisfied with $\lambda_1 \leqslant 0$, one can take some $\gamma > 0$ and add $-\lambda_1 u + \gamma u$ on both sides of (2.8); then Hypotheses 2.1 and 2.3 are fully satisfied, with $\beta(x)$ replaced by $\beta(x) - \lambda_1 + \gamma$ and $f(x,u)$ replaced by $f(x,u) - \lambda_1 u + \gamma u$.

We recall the following classical result (see e.g. Theorem II.1.3 in [6]):

Theorem 2.6. *Let X be a Banach space and let $A : D(A) \subset X \to X$ be the infinitesimal generator of a C^0-semigroup of linear operators $T(t)$, $t \in \mathbb{R}_+$. Consider the abstract Cauchy problem*

$$\begin{cases} \dot{u} = Au(t) + f(t), & t \in \mathbb{R}_+, \\ u(0) = u_0. \end{cases} \tag{2.9}$$

Assume that $u_0 \in D(A)$ and that either

(1) *$f \in C(\mathbb{R}_+, X)$ takes values in $D(A)$ and $AF \in C(\mathbb{R}_+, X)$, or*
(2) *$f \in C^1(\mathbb{R}_+, X)$.*

Then (2.9) *has a unique solution $u \in C^1(\mathbb{R}_+)$ with values in $D(A)$. The solution is given by*

$$u(t) = T(t)u_0 + \int_0^t T(t-s) f(s)\, ds. \tag{2.10}$$

Using Theorem 2.6, we rewrite Eq. (2.8) as an integral evolution equation in the space $Z_0 = H_0^1(\Omega) \times L^2(\Omega)$, namely

$$\big(u(t), v(t)\big) = \mathbf{T}_{\epsilon,0}(t)(u_0, v_0) + \int_0^t \mathbf{T}_{\epsilon,0}(t-p)\big(0, (1/\epsilon)\hat{f}\big(u(p)\big)\big)(dp \mid Z_0). \tag{2.11}$$

Eq. (2.11) is called the *mild formulation* of (2.8) and solutions of (2.11) are called *mild solutions* of (2.8). Note that by Proposition 2.4 the nonlinear operator $(u,v) \mapsto (0, \hat{f}(u))$ is Lipschitz continuous from Z_0 into itself. A classical Picard iteration argument shows that, if $(u_0, v_0) \in Z_0$, then (2.11) possesses a unique continuous maximal solution $(u(\cdot), v(\cdot)) : [0, t_{\max}[\to Z_0$ (see Theorem 4.3.4 and Proposition 4.3.7 in [2]). We thus obtain a local semiflow on Z_0. Notice that the solution $(u(\cdot), v(\cdot))$ of (2.11) also satisfies

$$\big(u(t), v(t)\big) = \mathbf{T}_{\epsilon,-1}(t)(u_0, v_0) + \int_0^t \mathbf{T}_{\epsilon,-1}(t-p)\big(0, (1/\epsilon)\hat{f}\big(u(p)\big)\big)(dp \mid Z_{-1}). \tag{2.12}$$

Therefore, it follows from Theorem 2.6 that $(u(\cdot), v(\cdot))$ is continuously differentiable into Z_{-1} and

$$(\partial_t \mid Z_{-1})\big(u(t), v(t)\big) = \mathbf{B}_{\epsilon,-1}\big(u(t), v(t)\big) + \big(0, (1/\epsilon)\hat{f}\big(u(t)\big)\big). \tag{2.13}$$

In particular, one has

$$\begin{cases} (\partial_t \mid H_0)u(t) = v(t), \\ \epsilon(\partial_t \mid H_{-1})v(t) = -v(t) - \mathbf{A}_{-1}u(t) + \hat{f}\big(u(t)\big). \end{cases} \tag{2.14}$$

Definition 2.7. A function $(u(\cdot), v(\cdot)) : \mathbb{R} \to Z_0$ is called a *full solution* of (2.11) iff, for every $s, t \in \mathbb{R}$, with $s \leqslant t$, one has

$$\big(u(t), v(t)\big) = \mathbf{T}_{\epsilon,0}(t-s)\big(u(s), v(s)\big) + \int_s^t \mathbf{T}_{\epsilon,0}(t-p)\big(0, (1/\epsilon)\hat{f}\big(u(p)\big)\big)(dp \mid Z_0).$$

Now we can state our first main result.

Theorem 2.8. *Assume that Hypotheses* 2.1 *and* 2.3 *are satisfied. Let* $\epsilon \in]0, 1]$ *be fixed and let* R *be a positive constant. Let* $(\bar{u}(\cdot), \bar{v}(\cdot)) : \mathbb{R} \to Z_0$ *be a bounded full solution of* (2.11)*, such that* $\sup_{t\in\mathbb{R}}(\|\bar{u}(t)\|^2_{H^1_0} + \epsilon\|\bar{v}(t)\|^2_{L^2}) \leqslant R$. *Assume that the first component* $\bar{u}(\cdot)$ *is uniformly continuous with modulus of continuity* $\omega(\cdot)$. *Then* $(\bar{u}(\cdot), \bar{v}(\cdot))$ *is continuous into* Z_1*, is continuously differentiable into* Z_0*, and*

$$(\partial_t \mid Z_0)\big(\bar{u}(t), \bar{v}(t)\big) = \mathbf{B}_{\epsilon,0}\big(\bar{u}(t), \bar{v}(t)\big) + \big(0, (1/\epsilon)\hat{f}\big(\bar{u}(t)\big)\big).$$

Moreover, there exists a positive constant $\tilde{R}_\epsilon$ *such that*

$$\sup_{t\in\mathbb{R}}\big(\big\|\mathbf{A}_0\bar{u}(t)\big\|^2_{L^2} + \big\|\bar{v}(t)\big\|^2_{H^1_0} + \epsilon\big\|(\partial_t \mid H_0)\bar{v}(t)\big\|^2_{L^2}\big) \leqslant \tilde{R}_\epsilon.$$

The constant $\tilde{R}_\epsilon$ *depends, besides* ϵ*, only on the constants in Hypotheses* 2.1 *and* 2.3*, on* R *and on* $\omega(\cdot)$.

We remind that a subset $\mathcal{I}$ of Z_0 is called *invariant* for the semiflow generated by (2.11) if for every $(u_0, v_0) \in \mathcal{I}$ there exists a full solution $(u(\cdot), v(\cdot))$ of (2.11) with $(u(0), v(0)) = (u_0, v_0)$ and $(u(t), v(t)) \in \mathcal{I}$ for all $t \in \mathbb{R}$.

Lemma 2.9. *(See Lemma 2.3 in [10].) If* $\mathcal{I}$ *is a compact invariant subset for the semiflow generated by* (2.11)*, then the set of all the full solutions of* (2.11) *in* $\mathcal{I}$ *is uniformly equicontinuous.*

Therefore, if $\mathcal{I}$ is a compact invariant subset for the semiflow generated by (2.11), then there exists a continuous, nondecreasing function $\omega : [0, 1] \to \mathbb{R}_+$, with $\omega(0) = 0$, such that, for every full solution $(u(\cdot), v(\cdot))$ of (2.11) in $\mathcal{I}$, one has

$$\big\|u(t) - u(s)\big\|_{H^1_0} \leqslant \omega\big(|t-s|\big), \quad t, s \in \mathbb{R},\ |t-s| \leqslant 1.$$

As a consequence of Theorem 2.8 and Lemma 2.9, one can easily prove the following corollary.

Corollary 2.10. *Assume that Hypotheses* 2.1 *and* 2.3 *are satisfied. Let* $\epsilon \in]0, 1]$ *be fixed. Let* $\mathcal{I}$ *be a compact invariant subset of the local semiflow generated by* (2.11) *in* Z_0. *Then* $\mathcal{I}$ *is a bounded subset of* Z_1.

Theorem 2.8 and Corollary 2.10 furnish estimates which depend heavily on ϵ. In many situations it is of interest to obtain estimates which are uniform in ϵ. To this end, we need to introduce the following hypothesis.

Hypothesis 2.11.

(1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is such that, for every $u \in \mathbb{R}$, $f(\cdot, u)$ is measurable and $f(\cdot, 0) \in L^2(\Omega)$;

(2) for a.e. $x \in \Omega$, $f(x, \cdot)$ is of class C^2, $\partial_u f(\cdot, 0) \in L^\infty(\Omega)$, $\partial_{uu} f(\cdot, 0) \in L^\infty(\Omega)$ and there exists a constant $C > 0$ such that

$$\left|\partial_{uu} f(x, u_1) - \partial_{uu} f(x, u_2)\right| \leqslant C|u_1 - u_2|.$$

Notice that Hypothesis 2.11 is a strengthening of Hypothesis 2.3. We have the following theorem.

Theorem 2.12. *Assume that Hypotheses* 2.1 *and* 2.11 *are satisfied. Let R be a positive constant, and for every $\epsilon \in]0, 1]$ let $(\bar{u}_\epsilon(\cdot), \bar{v}_\epsilon(\cdot)) : \mathbb{R} \to Z_0$ be a bounded full solution of* (2.11)*, such that $\sup_{t\in\mathbb{R}}(\|\bar{u}_\epsilon(t)\|^2_{H^1_0} + \epsilon\|\bar{v}_\epsilon(t)\|^2_{L^2}) \leqslant R$. Assume that, for every $\epsilon \in]0, 1]$, the first component $\bar{u}_\epsilon(\cdot)$ is uniformly continuous. Then there exists a positive constant $\tilde{R}$ such that, for every $\epsilon \in]0, 1]$,*

$$\sup_{t\in\mathbb{R}}\left(\left\|\mathbf{A}_0\bar{u}_\epsilon(t)\right\|^2_{L^2} + \left\|\bar{v}_\epsilon(t)\right\|^2_{H^1_0} + \epsilon\left\|(\partial_t \mid H_0)\bar{v}_\epsilon(t)\right\|^2_{L^2}\right) \leqslant \tilde{R}.$$

The constant $\tilde{R}$ depends only on the constants in Hypotheses 2.1 *and* 2.11 *and on R.*

One has also the following corollary.

Corollary 2.13. *Assume that Hypotheses* 2.1 *and* 2.11 *are satisfied. For every $\epsilon \in]0, 1]$, let $\mathcal{I}_\epsilon$ be a compact invariant subset of the local semiflow generated by* (2.11) *in Z_0. Assume that there exists $R > 0$ such that, for every $\epsilon \in]0, 1]$,*

$$\sup_{(u,v)\in\mathcal{I}_\epsilon}\left(\|u\|^2_{H^1_0} + \epsilon\|v\|^2_{L^2}\right) \leqslant R.$$

Then there exists $\tilde{R} > 0$ such that, for every $\epsilon \in]0, 1]$,

$$\sup_{(u,v)\in\mathcal{I}_\epsilon}\left(\|\mathbf{A}_0 u\|^2_{L^2} + \|v\|^2_{H^1_0}\right) \leqslant \tilde{R}.$$

The constant $\tilde{R}$ depends only on the constants in Hypotheses 2.1 *and* 2.11 *and on R.*

3. The linear evolution system

Let $\epsilon \in]0, 1]$ be fixed, let R be a positive constant and let $(\bar{u}(\cdot), \bar{v}(\cdot)) : \mathbb{R} \to Z_0$ be a bounded full solution of (2.11), such that $\sup_{t\in\mathbb{R}}(\|\bar{u}(t)\|^2_{H^1_0} + \epsilon\|\bar{v}(t)\|^2_{L^2}) \leqslant R$. In this section we study the evolution system generated by the linear equation

$$\partial_t\big(v(t), w(t)\big) = \mathbf{B}_{\epsilon,-1}\big(v(t), w(t)\big) + \Big(0, (1/\epsilon)\big(-\theta + \widehat{\partial_u f}\big(\bar{u}(t)\big)\big) \cdot v(t)\Big)$$

in the space Z_{-1}, where θ is a positive constant.

Definition 3.1. Let X be a Banach space and let $J \subset \mathbb{R}$ be an interval. A two-parameter family of bounded linear operators $U(t, s)$, $s, t \in J$, $s \leqslant t$, is called an *evolution system* on X iff the following conditions are satisfied:

(1) $U(s, s) = I$, $U(t, r)U(r, s) = U(t, s)$ for $s, r, t \in J$, $s \leqslant r \leqslant t$;
(2) $(t, s) \mapsto U(t, s)$ is strongly continuous into $\mathcal{L}(X)$ for $s, t \in J$, $s \leqslant t$.

We recall the fundamental theorem of Kato (see [12]), which provides sufficient conditions for the existence of an evolution system. Let X be a Banach space. We denote by $\mathcal{G}(X)$ the set of all infinitesimal generators of C^0-semigroups of linear operators on X.

Definition 3.2. Let X be a Banach space and let $J \subset \mathbb{R}$ be an interval. A one-parameter family of linear operators $A(t) \in \mathcal{G}(X)$, $t \in J$, is called *stable* iff there are constants $M > 0$, $\beta \in \mathbb{R}$ (called the *constants of stability*) such that

$$\left\| \prod_{j=1}^{k} \bigl(A(t_j) + \lambda\bigr)^{-1} \right\|_{\mathcal{L}(X)} \leqslant M(\lambda - \beta)^{-k}, \quad \lambda > \beta,$$

for any finite family $(t_j)_{j=1}^k$ of points of J, with $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_k$, $k \in \mathbb{N}$.

Theorem 3.3. *(See Theorem 6.1 in [12].) Let X and Y be Banach spaces, such that Y is densely and continuously embedded in X. Let $A(t)$, $t \in \mathcal{G}(X)$, be a family of linear operators such that:*

(1) *$(A(t))_{t \in J}$ is stable with constants M and β;*
(2) *there is a family $(S(t))_{t \in J}$ of isomorphisms of Y to X such that $S(\cdot)$ is strongly continuously differentiable into $\mathcal{L}(Y, X)$ and*

$$S(t)A(t)S(t)^{-1} = A(t) + B(t), \quad B(t) \in \mathcal{L}(X),$$

where $B(\cdot)$ is strongly continuous into $\mathcal{L}(X)$;
(3) *$Y \subset D(A(t))$, so that $A(t) \in \mathcal{L}(Y, X)$ for $t \in J$, and the map $t \mapsto A(t)$ is norm continuous into $\mathcal{L}(Y, X)$.*

Under these conditions, there exists a unique evolution system $U(t, s)$ on X, defined for $s, t \in J$, $s \leqslant t$, with the following properties:

(1) *$\|U(t, s)\|_{\mathcal{L}(X)} \leqslant Me^{\beta(t-s)}$;*
(2) *$U(t, s)Y \subset Y$ and $\|U(t, s)|_Y\|_{\mathcal{L}(Y)} \leqslant \tilde{M}e^{\tilde{\beta}(t-s)}$ for some constants $\tilde{M} > 0$, $\tilde{\beta} \in \mathbb{R}$;*
(3) *the map $(s, t) \mapsto U(t, s)|_Y$ is strongly continuous in $\mathcal{L}(Y)$ for $s, t \in J$, $s \leqslant t$;*
(4) *for each fixed $y \in Y$ and $t \in J$, the mapping $s \mapsto U(t, s)y$ is continuously differentiable in X and $(d/ds)U(t, s)y = -U(t, s)A(s)y$, $s \leqslant t$;*
(5) *for each fixed $y \in Y$ and $s \in J$, the mapping $t \mapsto U(t, s)y$ is continuously differentiable in X and $(d/dt)U(t, s)y = A(t)U(t, s)y$, $s \leqslant t$.*

In order to exploit Kato's theorem, we need to introduce some notation. For $\kappa \in \mathbb{R}$ and $\theta \geqslant 0$, define $\mathbf{A}_\kappa[\theta] := \mathbf{A}_\kappa + \theta\mathbf{I} : H_{\kappa+2} \to H_\kappa$. For $\epsilon \in]0, 1]$, $\kappa \in \mathbb{R}$ and $\theta \geqslant 0$, define the linear operator $\mathbf{B}_{\epsilon,\kappa}[\theta] : Z_{\kappa+1} \to Z_\kappa$ by

$$\mathbf{B}_{\epsilon,\kappa}[\theta](u, v) := \bigl(v, -(1/\epsilon)\bigl(v + \mathbf{A}_\kappa[\theta]u\bigr)\bigr), \quad (u, v) \in Z_{\kappa+1}.$$

It follows that $\mathbf{B}_{\epsilon,\kappa}[\theta]$ is the infinitesimal generator of a C^0-semigroup $\mathbf{T}_{\epsilon,\kappa}[\theta](t)$, $t \in [0, +\infty[$, on Z_κ.

For $t \in \mathbb{R}$, define the operator $\mathbf{C}_{\epsilon,-1}(t) : Z_{-1} \to Z_{-1}$ by

$$\mathbf{C}_{\epsilon,-1}(t)(u, v) := \bigl(0, (1/\epsilon)\widehat{\partial_u f}\bigl(\bar{u}(t)\bigr) \cdot u\bigr).$$

Notice that, by (2.6), the mapping $t \mapsto \mathbf{C}_{\epsilon,-1}(t)$ is norm continuous into $\mathcal{L}(Z_{-1})$. Moreover, by (2.3), $\mathbf{C}_{\epsilon,-1}(t)$ maps Z_0 into itself. Setting $\mathbf{C}_{\epsilon,0}(t) := \mathbf{C}_{\epsilon,-1}(t)|_{Z_0}$, we get from (2.4) that the mapping $t \mapsto \mathbf{C}_{\epsilon,0}(t)$ is norm continuous into $\mathcal{L}(Z_0)$.

Proposition 3.4. *Let $\theta \geqslant 0$. Set $X := Z_{-1}$, $Y := Z_0$, $A(t) := \mathbf{B}_{\epsilon,-1}[\theta] + \mathbf{C}_{\epsilon,-1}(t)$ and $S(t) := (\mathbf{B}_{\epsilon,-1}[\theta])^{-1}$, $t \in \mathbb{R}$. Then the assumptions of Theorem* 3.3 *are satisfied.*

Proof. The stability of the family $A(t)$ follows from Proposition 3.5 in [12]. The norm continuity of the mapping $t \mapsto A(t)$ is a consequence of (2.6). In order to conclude, we shall compute explicitly $S(t)A(t)S(t)^{-1}$. We have that $S(t)A(t)S(t)^{-1} = A(t) + B(t)$, where

$$B(t) = -\mathbf{C}_{\epsilon,-1}(t) + \mathbf{B}_{\epsilon,-1}[\theta]\mathbf{C}_{\epsilon,-1}(t)\big(\mathbf{B}_{\epsilon,-1}[\theta]\big)^{-1}. \tag{3.1}$$

The first addendum in (3.1) is strongly continuous into $\mathcal{L}(X)$ by (2.6). Concerning the second summand, an explicit computation shows that

$$\big(\mathbf{B}_{\epsilon,-1}[\theta]\big)^{-1}(u,v) = \big(-\big(\mathbf{A}_{-1}[\theta]\big)^{-1}(\epsilon v + u), u\big).$$

It follows that

$$\begin{aligned}&\mathbf{B}_{\epsilon,-1}[\theta]\mathbf{C}_{\epsilon,-1}(t)\big(\mathbf{B}_{\epsilon,-1}[\theta]\big)^{-1}(u,v)\\ &\quad= \big(-(1/\epsilon)\widehat{\partial_u f}\big(\bar{u}(t)\big)\cdot\big(\mathbf{A}_{-1}[\theta]\big)^{-1}(\epsilon v + u), \big(1/\epsilon^2\big)\widehat{\partial_u f}\big(\bar{u}(t)\big)\cdot\big(\mathbf{A}_{-1}[\theta]\big)^{-1}(\epsilon v + u)\big).\end{aligned}$$

Now it follows from (2.3) and (2.4) that the second addendum in (3.1) is strongly continuous into $\mathcal{L}(X)$. □

From now on, we denote by $\mathbf{U}_{\epsilon,-1}[\theta](t,s)$ the evolution family generated by $\mathbf{B}_{\epsilon,-1}[\theta] + \mathbf{C}_{\epsilon,-1}(t)$ in Z_{-1} and by $\mathbf{U}_{\epsilon,0}[\theta](t,s)$ its restriction to Z_0. Our next goal is to obtain suitable decay estimates for $\mathbf{U}_{\epsilon,-1}[\theta](t,s)$ and $\mathbf{U}_{\epsilon,0}[\theta](t,s)$. To this end, we need to introduce some more notation.

For $\theta \geqslant 0$, we define the following scalar product in $H_0^1(\Omega)$:

$$\begin{aligned}\langle u, v\rangle_{H_0^1[\theta]} :=& \int_\Omega A(x)\nabla u(x)\cdot\nabla v(x)\,dx\\ &+ \int_\Omega \beta(x)u(x)v(x)\,dx + \int_\Omega \theta u(x)v(x)\,dx, \quad u, v \in H^1(\Omega).\end{aligned} \tag{3.2}$$

We denote by $\|\cdot\|_{H_0^1[\theta]}$ the corresponding norm. Moreover, we denote by $\langle\cdot,\cdot\rangle_{H^{-1}[\theta]}$ the scalar product in $H^{-1}(\Omega)$ dual to $\langle\cdot,\cdot\rangle_{H_0^1[\theta]}$, and by $\|\cdot\|_{H^{-1}[\theta]}$ the corresponding norm. We have the following estimates:

$$\left(\frac{\lambda_1}{\theta + \lambda_1}\right)^{1/2}\|\cdot\|_{H_0^1[\theta]} \leqslant \|\cdot\|_{H_0^1} \leqslant \|\cdot\|_{H_0^1[\theta]} \tag{3.3}$$

and

$$\left(\frac{\lambda_1}{\theta + \lambda_1}\right)^{1/2}\|\cdot\|_{H^{-1}} \leqslant \|\cdot\|_{H^{-1}[\theta]} \leqslant \|\cdot\|_{H^{-1}}. \tag{3.4}$$

Moreover,

$$\|u\|^2_{H^1[\theta]} \geqslant (\lambda_1 + \theta)\|u\|^2_{L^2}, \quad u \in H_0^1(\Omega), \tag{3.5}$$

and

$$\|u\|_{L^2}^2 \geqslant (\lambda_1 + \theta)\|u\|_{H^{-1}[\theta]}^2, \quad u \in L^2(\Omega). \tag{3.6}$$

Notice also that

$$\langle u, v\rangle_{H_0^1[\theta]} = \langle \mathbf{A}_0[\theta]u, v\rangle_{L^2}, \quad u \in D\big(\mathbf{A}_0[\theta]\big),\ v \in H_0^1(\Omega), \tag{3.7}$$

and

$$\langle u, v\rangle_{L^2} = \langle \mathbf{A}_{-1}[\theta]u, v\rangle_{H^{-1}[\theta]}, \quad u \in D\big(\mathbf{A}_{-1}[\theta]\big),\ v \in L^2(\Omega). \tag{3.8}$$

For $\theta \geqslant 0$, we define the following norms in Z_0 and Z_{-1} respectively:

$$\big\|(u, v)\big\|_{Z_{\epsilon,0}[\theta]} := \|u\|_{H_0^1[\theta]} + \epsilon^{1/2}\|v\|_{L^2}, \qquad \big\|(u, v)\big\|_{Z_{\epsilon,-1}[\theta]} := \|u\|_{L^2} + \epsilon^{1/2}\|v\|_{H^{-1}[\theta]}.$$

For $\theta \geqslant 0$ and $\tau \in \mathbb{R}$, we define also the following bilinear form in $H_0^1(\Omega)$:

$$\begin{aligned}\langle u, v\rangle_{H_0^1[\theta,\tau]} := {} & \int_\Omega A(x)\nabla u(x)\cdot\nabla v(x)\,dx + \int_\Omega \beta(x)u(x)v(x)\,dx \\ & + \int_\Omega \theta u(x)v(x)\,dx - \int_\Omega \widehat{\partial_u f}\big(\bar{u}(\tau)\big)(x)u(x)v(x)\,dx, \quad u, v \in H^1(\Omega).\end{aligned} \tag{3.9}$$

We shall see in a moment that, for sufficiently large θ, $\langle\cdot,\cdot\rangle_{H_0^1[\theta,\tau]}$ is in fact a scalar product. We denote by $\|\cdot\|_{H_0^1[\theta,\tau]}$ the corresponding norm. Moreover, we denote by $\langle\cdot,\cdot\rangle_{H^{-1}[\theta,\tau]}$ the scalar product in $H^{-1}(\Omega)$ dual to $\langle\cdot,\cdot\rangle_{H_0^1[\theta,\tau]}$, and by $\|\cdot\|_{H^{-1}[\theta,\tau]}$ the corresponding norm. For $\kappa = 0, -1$, define $\mathbf{A}_\kappa[\theta,\tau] := \mathbf{A}_\kappa[\theta] - \widehat{\partial_u f}(\bar{u}(\tau))$ and notice that

$$\langle u, v\rangle_{H_0^1[\theta,\tau]} = \langle \mathbf{A}_0[\theta,\tau]u, v\rangle_{L^2}, \quad u \in D\big(\mathbf{A}_0[\theta,\tau]\big),\ v \in H_0^1(\Omega), \tag{3.10}$$

and

$$\langle u, v\rangle_{L^2} = \langle \mathbf{A}_{-1}[\theta,\tau]u, v\rangle_{H^{-1}[\theta,\tau]}, \quad u \in D\big(\mathbf{A}_{-1}[\theta,\tau]\big),\ v \in L^2(\Omega). \tag{3.11}$$

We need the following lemma.

Lemma 3.5. *For every ρ, with $0 < \rho < 1$, there exists $\theta_\rho \geqslant 0$ such that, for all $\theta \geqslant \theta_\rho$ and $\tau \in \mathbb{R}$,*

$$(1-\rho)^{1/2}\|\cdot\|_{H_0^1[\theta]} \leqslant \|\cdot\|_{H_0^1[\theta,\tau]} \leqslant (1+\rho)^{1/2}\|\cdot\|_{H_0^1[\theta]} \tag{3.12}$$

and

$$(1-\rho)^{1/2}\|\cdot\|_{H^{-1}[\theta,\tau]} \leqslant \|\cdot\|_{H^{-1}[\theta]} \leqslant (1+\rho)^{1/2}\|\cdot\|_{H^{-1}[\theta,\tau]}. \tag{3.13}$$

The constant θ_ρ, besides ρ, depends only on R and on the constants in Hypotheses 2.1 *and* 2.3.

Proof. First we observe that for every $\nu > 0$ there is a $C_\nu > 0$ with

$$\int_\Omega \left|\widehat{\partial_u f}\left(\bar{u}(\tau)\right)(x)\right|\left|u(x)\right|^2 dx \leqslant \nu \int_\Omega \left|\nabla u(x)\right|^2 dx + C_\nu \int_\Omega \left|u(x)\right|^2 dx$$

for all $u \in H_0^1(\Omega)$. The constant C_ν, besides ν, depends on the constants in Hypotheses 2.1 and 2.3. It follows that

$$\|u\|^2_{H_0^1[\theta,\tau]} \leqslant \|u\|^2_{H_0^1[\theta]} + \nu\|u\|^2_{H_0^1} + C_\nu\|u\|^2_{L^2} \leqslant \|u\|^2_{H_0^1[\theta]} + \nu\|u\|^2_{H_0^1[\theta]} + \frac{C_\nu}{\theta+\lambda_1}\|u\|^2_{H_0^1[\theta]}.$$

On the other hand,

$$\|u\|^2_{H_0^1[\theta,\tau]} \geqslant \|u\|^2_{H_0^1[\theta]} - \nu\|u\|^2_{H_0^1} - C_\nu\|u\|^2_{L^2} \geqslant \|u\|^2_{H_0^1[\theta]} - \nu\|u\|^2_{H_0^1[\theta]} - \frac{C_\nu}{\theta+\lambda_1}\|u\|^2_{H_0^1[\theta]}.$$

Choosing first $\nu = \rho/2$ and then θ_ρ such that $C_\nu/(\theta_\rho + \lambda_1) \leqslant \rho/2$, we obtain (3.12). Estimates (3.13) follow from (3.12) and a duality argument. □

For $\theta \geqslant 0$ and $\tau \in \mathbb{R}$, we denote by $\mathbf{T}_{\epsilon,0}[\theta,\tau](t)$ (resp. by $\mathbf{T}_{\epsilon,-1}[\theta,\tau](t)$) the semigroup generated by $\mathbf{B}_{\epsilon,0}[\theta] + \mathbf{C}_{\epsilon,0}(\tau)$ in Z_0 (resp. by $\mathbf{B}_{\epsilon,-1}[\theta] + \mathbf{C}_{\epsilon,-1}(\tau)$ in Z_{-1}).

Lemma 3.6. *Let $s \in \mathbb{R}$ and let $(v_s, w_s) \in Z_{-1}$. Set $(v(t), w(t)) := \mathbf{U}_{\epsilon,-1}[\theta](t,s)(v_s, w_s)$, $t \geqslant s$. Then, for $s \leqslant \tau \leqslant t$,*

$$\begin{aligned}\left(v(t), w(t)\right) = {} & \mathbf{T}_{\epsilon,-1}[\theta,\tau](t-\tau)\left(v(\tau), w(\tau)\right) \\ & + \int_\tau^t \mathbf{T}_{\epsilon,-1}[\theta,\tau](t-p)\left(\mathbf{C}_{\epsilon,-1}(p) - \mathbf{C}_{\epsilon,-1}(\tau)\right)\left(v(p), w(p)\right)(dp \mid Z_{-1}).\end{aligned} \tag{3.14}$$

Proof. We suppose first that $(v_s, w_s) \in Z_0$. By property (5) in Theorem 3.3 we have that $(v(\cdot), w(\cdot))$ is continuously differentiable into Z_{-1}, continuous into Z_0, and satisfies

$$\begin{aligned}(\partial_t \mid Z_{-1})\left(v(t), w(t)\right) &= \left(\mathbf{B}_{\epsilon,-1}[\theta] + \mathbf{C}_{\epsilon,-1}(t)\right)\left(v(t), w(t)\right) \\ &= \left(\mathbf{B}_{\epsilon,-1}[\theta] + \mathbf{C}_{\epsilon,-1}(\tau)\right)\left(v(t), w(t)\right) + \left(\mathbf{C}_{\epsilon,-1}(t) - \mathbf{C}_{\epsilon,-1}(\tau)\right)\left(v(t), w(t)\right).\end{aligned}$$

Since the mapping $t \mapsto (\mathbf{C}_{\epsilon,-1}(t) - \mathbf{C}_{\epsilon,-1}(\tau))(v(t), w(t))$ is continuous into $Z_0 = D(\mathbf{B}_{\epsilon,-1}[\theta] + \mathbf{C}_{\epsilon,-1}(\tau))$, the conclusion follows from Theorem 2.6. If $(v_s, w_s) \in Z_{-1}$, the conclusion follows from a density argument. □

Now we are ready to state and prove the desired decay estimates for $\mathbf{U}_{\epsilon,-1}[\theta](t,s)$ and $\mathbf{U}_{\epsilon,0}[\theta](t,s)$.

Proposition 3.7. *Assume that $\bar{u}(\cdot)$ is uniformly continuous. Then there exist $\bar{\theta}_0 > 0$, $\delta > 0$, $M > 0$ such that, for all $\theta \geqslant \bar{\theta}_0$,*

$$\left\|\mathbf{U}_{\epsilon,-1}[\theta](t,s)\right\|_{\mathcal{L}(Z_{\epsilon,-1}[\theta])} \leqslant M e^{-\delta(t-s)}, \quad t \geqslant s, \tag{3.15}$$

and

$$\left\|\mathbf{U}_{\epsilon,0}[\theta](t,s)\right\|_{\mathcal{L}(Z_{\epsilon,0}[\theta])} \leqslant Me^{-\delta(t-s)}, \quad t \geqslant s. \tag{3.16}$$

The constant δ depends only on the constants in Hypothesis 2.1. *The constants $\bar{\theta}_0$ and M depend on R, on the modulus of continuity ω of $\bar{u}(\cdot)$ and on the constants in Hypotheses* 2.1 *and* 2.3.

Proof. The proof is similar to, and inspired by, the proof of inequality (3.43) in [10]. We begin by proving (3.15). Let $\theta \geqslant \theta_{1/2}$, where $\theta_{1/2}$ is given by Lemma 3.5, let $s \in \mathbb{R}$ and let $(v_s, w_s) \in Z_{-1}$. Set $(v(t), w(t)) := \mathbf{U}_{\epsilon,-1}[\theta](t,s)(v_s, w_s)$, $t \geqslant s$. By Lemma 3.6 above, we have, for $s \leqslant \tau \leqslant t$, that

$$\begin{aligned}\big(v(t), w(t)\big) &= \mathbf{T}_{\epsilon,-1}[\theta,\tau](t-\tau)\big(v(\tau), w(\tau)\big)\\ &\quad + \int_{\tau}^{t} \mathbf{T}_{\epsilon,-1}[\theta,\tau](t-p)\big(\mathbf{C}_{\epsilon,-1}(p) - \mathbf{C}_{\epsilon,-1}(\tau)\big)\big(v(p), w(p)\big)(dp \mid Z_{-1}).\end{aligned} \tag{3.17}$$

Let δ be a positive constant, with $\delta \leqslant \min\{1/2, \lambda_1/2\}$, and let η be a positive constant to be fixed later. For $j \in \mathbb{N}_0$, we define the intervals $I_j := [s + j\eta, s + (j+1)\eta]$. For $j \in \mathbb{N}_0$, we introduce the following family of energy functionals on Z_{-1}:

$$E_{\theta,j}(v,w) := \frac{1}{2}\epsilon\|\delta v + w\|^2_{H^{-1}[\theta,s+j\eta]} + \frac{1}{2}\|v\|^2_{L^2} + \frac{1}{2}\big(\epsilon\delta^2 - \delta\big)\|v\|^2_{H^{-1}[\theta,s+j\eta]}. \tag{3.18}$$

Moreover, we define

$$E_{\theta}(v,w) := \frac{1}{2}\epsilon\|\delta v + w\|^2_{H^{-1}[\theta]} + \frac{1}{2}\|v\|^2_{L^2} + \frac{1}{2}\big(\epsilon\delta^2 - \delta\big)\|v\|^2_{H^{-1}[\theta]}. \tag{3.19}$$

Since $\epsilon \in]0,1]$ and $\delta \leqslant \min\{1/2, \lambda_1/2\}$, a direct computation using (3.6) shows that, for all $\theta \geqslant \theta_{1/2}$,

$$\frac{1}{4}\big\|(v,w)\big\|_{Z_{\epsilon,-1}[\theta]} \leqslant E_\theta(v,w) \leqslant \frac{3}{4}\big\|(v,w)\big\|_{Z_{\epsilon,-1}[\theta]}, \quad (v,w) \in Z_{-1}. \tag{3.20}$$

Moreover, by Lemma 3.5, for every ρ, with $0 < \rho \leqslant 1/2$, there exists $\theta_\rho \geqslant \theta_{1/2}$ such that, for all $\theta \geqslant \theta_\rho$ and all $j \in \mathbb{N}_0$,

$$(1-\rho)E_{\theta,j}(v,w) \leqslant E_\theta(v,w) \leqslant (1+\rho)E_{\theta,j}(v,w), \quad (v,w) \in Z_{-1}. \tag{3.21}$$

An elementary, but quite tedious computation, using (3.17) with $\tau = s + j\eta$, Theorem 2.6 in [15], and (3.11), shows that the mapping $t \mapsto E_{\theta,j}(v(t), w(t))$ is differentiable on I_j, and

$$\begin{aligned}&\frac{d}{dt}E_{\theta,j}\big(v(t), w(t)\big) + 2\delta E_{\theta,j}\big(v(t), w(t)\big)\\ &\quad = (2\epsilon\delta - 1)\big\|\delta v(t) + w(t)\big\|^2_{H^{-1}(\theta,s+j\eta)}\\ &\qquad + \big\langle \delta v(t) + w(t), \big(\widehat{\partial_u f}\big(\bar{u}(t)\big) - \widehat{\partial_u f}\big(\bar{u}(s+j\eta)\big)\big)v(t)\big\rangle_{H^{-1}(\theta,s+j\eta)}.\end{aligned} \tag{3.22}$$

Take ρ, with $0 < \rho \leqslant 1/2$, and take $\theta \geqslant \theta_\rho$. Using Cauchy–Schwartz inequality, inequalities (3.4), (3.13) and (2.6), and the fact that $(2\epsilon\delta - 1) \leqslant -1/2$, we obtain

$$\begin{aligned}
&\frac{d}{dt}E_{\theta,j}\big(v(t),w(t)\big)+2\delta E_{\theta,j}\big(v(t),w(t)\big)\\
&\quad\leqslant\frac{1}{2}\big\|\big(\widehat{\partial_u f}\big(\bar u(t)\big)-\widehat{\partial_u f}\big(\bar u(s+j\eta)\big)\big)v(t)\big\|^2_{H^{-1}(\theta,s+j\eta)}\\
&\quad\leqslant\frac{1}{2}(1-\rho)^{-1}\big\|\big(\widehat{\partial_u f}\big(\bar u(t)\big)-\widehat{\partial_u f}\big(\bar u(s+j\eta)\big)\big)v(t)\big\|^2_{H^{-1}}\\
&\quad\leqslant\frac{1}{2}(1-\rho)^{-1}\big\|\widehat{\partial_u f}\big(\bar u(t)\big)-\widehat{\partial_u f}\big(\bar u(s+j\eta)\big)\big\|^2_{\mathcal{L}(L^2,H^{-1})}\big\|v(t)\big\|^2_{L^2}\\
&\quad\leqslant\frac{1}{2}(1-\rho)^{-1}\tilde C\big(1+2R^{\alpha}\big)^2\big\|\bar u(t)-\bar u(s+j\eta)\big\|^{2\beta}_{H^1_0}\big\|v(t)\big\|^2_{L^2}\\
&\quad\leqslant\frac{1}{2}(1-\rho)^{-1}\tilde C\big(1+2R^{\alpha}\big)^2\omega(\eta)^{2\beta}\big\|v(t)\big\|^2_{L^2}\\
&\quad\leqslant(1-\rho)^{-1}\tilde C\big(1+2R^{\alpha}\big)^2\omega(\eta)^{2\beta}E_{\theta,j}\big(v(t),w(t)\big).
\end{aligned}\tag{3.23}$$

Now, recalling that $\rho\leqslant(1/2)$, we choose η in such a way that

$$2\tilde C\big(1+2R^{\alpha}\big)^2\omega(\eta)^{2\beta}\leqslant\delta.$$

With this choice, we have

$$\frac{d}{dt}E_{\theta,j}\big(v(t),w(t)\big)+\delta E_{\theta,j}\big(v(t),w(t)\big)\leqslant 0.\tag{3.24}$$

It follows that, for $t\in I_j$,

$$E_{\theta,j}\big(v(t),w(t)\big)\leqslant e^{-\delta(t-(s+j\eta))}E_{\theta,j}\big(v(s+j\eta),w(s+j\eta)\big).\tag{3.25}$$

Iterating inequality (3.25) and taking into account (3.21), we obtain, for $j\in\mathbb{N}_0$ and $t\in I_j$, that

$$E_{\theta}\big(v(t),w(t)\big)\leqslant\left(\frac{1+\rho}{1-\rho}\right)^{j+1}e^{-\delta(t-s)}E_{\theta}(v_s,w_s).\tag{3.26}$$

We are still free to choose $\rho\in\,]0,1/2]$. At this point, we observe that $t-s\geqslant j\eta$. Therefore, we choose ρ in such a way that

$$\left(\frac{1+\rho}{1-\rho}\right)e^{-\delta\eta/2}\leqslant 1.$$

With this choice, we obtain that, for $\theta\geqslant\theta_\rho$,

$$E_{\theta}\big(v(t),w(t)\big)\leqslant\left(\frac{1+\rho}{1-\rho}\right)e^{-\delta/2(t-s)}E_{\theta}(v_s,w_s),\quad t\geqslant s.\tag{3.27}$$

Finally, putting together (3.20) and (3.27), we obtain (3.15).

The proof of (3.16) is completely analogous. One needs only to replace $E_{\theta,j}(v,w)$ by

$$\tilde E_{\theta,j}(v,w):=\frac{1}{2}\epsilon\|\delta v+w\|^2_{L^2}+\frac{1}{2}\|v\|^2_{H^1_0[\theta,s+j\eta]}+\frac{1}{2}\big(\epsilon\delta^2-\delta\big)\|v\|^2_{L^2},\tag{3.28}$$

and $E_\theta(v,w)$ by

$$\tilde{E}_\theta(v,w) := \frac{1}{2}\epsilon\|\delta v + w\|^2_{L^2} + \frac{1}{2}\|v\|^2_{H^1_0[\theta]} + \frac{1}{2}\big(\epsilon\delta^2 - \delta\big)\|v\|^2_{L^2}, \tag{3.29}$$

where now $(v,w)\in Z_0$. The details are left to the reader. □

4. Proof of Theorem 2.8

Let R be a positive constant. Throughout this section we fix $\epsilon \in]0,1]$ and we denote by $(\bar{u}(\cdot),\bar{v}(\cdot)) : \mathbb{R} \to Z_0$ a fixed bounded full solution of (2.11), such that $\sup_{t\in\mathbb{R}}(\|\bar{u}(t)\|^2_{H^1_0} + \epsilon\|\bar{v}(t)\|^2_{L^2}) \leqslant R$.

As we have seen in Section 2 (see (2.14)), $(\bar{u}(\cdot),\bar{v}(\cdot))$ is continuously differentiable into Z_{-1} and

$$\begin{cases} (\partial_t \mid H_0)\bar{u}(t) = \bar{v}(t), \\ \epsilon(\partial_t \mid H_{-1})\bar{v}(t) = -\bar{v}(t) - \mathbf{A}_{-1}\bar{u}(t) + \hat{f}\big(\bar{u}(t)\big). \end{cases}$$

Set $\bar{w}(t) := (\partial_t \mid H_{-1})\bar{v}(t)$, $t\in\mathbb{R}$. Using (2.7) we see that $(\bar{v}(\cdot),\bar{w}(\cdot))$ is continuous into Z_{-1} and continuously differentiable into Z_{-2}, and

$$\begin{cases} (\partial_t \mid H_{-1})\bar{v}(t) = \bar{w}(t), \\ \epsilon(\partial_t \mid H_{-2})\bar{w}(t) = -\bar{w}(t) - \mathbf{A}_{-2}\bar{v}(t) + \widehat{\partial_u f}\big(\bar{u}(t)\big)\cdot\bar{v}(t). \end{cases}$$

Since the mapping $t \mapsto (0, \widehat{\partial_u f}(\bar{u}(t))\cdot\bar{v}(t))$ is continuous into $Z_{-1} = D(\mathbf{B}_{\epsilon,-2})$, it follows from Theorem 2.6 that, for $s,t\in\mathbb{R}$, with $s\leqslant t$, $(\bar{v}(\cdot),\bar{w}(\cdot))$ satisfies the equality

$$\begin{aligned}\big(\bar{v}(t),\bar{w}(t)\big) &= \mathbf{T}_{\epsilon,-2}(t-s)\big(\bar{v}(s),\bar{w}(s)\big) \\ &\quad + \int_s^t \mathbf{T}_{\epsilon,-2}(t-p)\big(0,(1/\epsilon)\widehat{\partial_u f}\big(\bar{u}(p)\big)\cdot\bar{v}(p)\big)(dp \mid Z_{-2}).\end{aligned}$$

Finally, since $(\bar{v}(\cdot),\bar{w}(\cdot))$ is continuous into Z_{-1}, it follows that $(\bar{v}(\cdot),\bar{w}(\cdot))$ satisfies the equality

$$\begin{aligned}\big(\bar{v}(t),\bar{w}(t)\big) &= \mathbf{T}_{\epsilon,-1}(t-s)\big(\bar{v}(s),\bar{w}(s)\big) \\ &\quad + \int_s^t \mathbf{T}_{\epsilon,-1}(t-p)\big(0,(1/\epsilon)\widehat{\partial_u f}\big(\bar{u}(p)\big)\cdot\bar{v}(p)\big)(dp \mid Z_{-1}).\end{aligned} \tag{4.1}$$

Notice that, for $s\in\mathbb{R}$ fixed, the function $(\bar{v}(\cdot),\bar{w}(\cdot))$ is the unique mild solution of (4.1) on $[s,+\infty[$. This is a consequence of (2.5) and (2.6). Now we want to give another representation of $(\bar{v}(\cdot),\bar{w}(\cdot))$, by mean of a *variation of constant formula* involving the evolution system $\mathbf{U}_{\epsilon,-1}[\theta](t,s)$ introduced in Section 3.

We need the following lemma.

Lemma 4.1. *Let $h(\cdot):\mathbb{R}\to L^2(\Omega)$ be a continuous function and let θ be a positive constant. Let $(v_s,w_s)\in Z_{-1}$ and let $(\tilde{v}(\cdot),\tilde{w}(\cdot)) : [0,+\infty[\to Z_{-1}$ be the unique solution of*

$$\begin{aligned}
&\big(v(t), w(t)\big) \\
&\quad = \mathbf{T}_{\epsilon,-1}(t-s)(v_s, w_s) \\
&\qquad + \int_s^t \mathbf{T}_{\epsilon,-1}(t-p)\big(\big(0, -(\theta/\epsilon)v(p)\big) + \mathbf{C}_{\epsilon,-1}(p)\big(v(p), w(p)\big) + \big(0, h(p)\big)\big)(dp \mid Z_{-1}). \quad (4.2)
\end{aligned}$$

Then

$$\big(\tilde{v}(t), \tilde{w}(t)\big) = \mathbf{U}_{\epsilon,-1}[\theta](t,s)(v_s, w_s) + \int_s^t \mathbf{U}_{\epsilon,-1}[\theta](t,p)\big(0, h(p)\big)(dp \mid Z_{-1}). \quad (4.3)$$

Proof. We suppose first that $(v_s, w_s) \in Z_0$. Define

$$\big(\check{v}(t), \check{w}(t)\big) = \mathbf{U}_{\epsilon,-1}[\theta](t,s)(v_s, w_s) + \int_s^t \mathbf{U}_{\epsilon,-1}[\theta](t,p)\big(0, h(p)\big)(dp \mid Z_{-1}).$$

By Theorem 7.1 in [12], we have that $(\check{v}(\cdot), \check{w}(\cdot))$ is continuously differentiable into Z_{-1}, continuous into Z_0, and satisfies

$$\begin{aligned}
(\partial_t \mid Z_{-1})\big(\check{v}(t), \check{w}(t)\big) &= \big(\mathbf{B}_{\epsilon,-1}[\theta] + \mathbf{C}_{\epsilon,-1}(t)\big)\big(\check{v}(t), \check{w}(t)\big) + \big(0, h(t)\big) \\
&= \mathbf{B}_{\epsilon,-1}\big(\check{v}(t), \check{w}(t)\big) - \big(0, (\theta/\epsilon)\check{v}(t)\big) + \mathbf{C}_{\epsilon,-1}(t)\big(\check{v}(t), \check{w}(t)\big) + \big(0, h(t)\big).
\end{aligned}$$

Since the mapping $t \mapsto -(0, (\theta/\epsilon)\check{v}(t)) + \mathbf{C}_{\epsilon,-1}(t)(\check{v}(t), \check{w}(t)) + (0, h(t))$ is continuous into $Z_0 = D(\mathbf{B}_{\epsilon,-1})$, it follows from Theorem 2.6 that

$$\begin{aligned}
\big(\check{v}(t), \check{w}(t)\big) &= \mathbf{T}_{\epsilon,-1}(t-s)(v_s, w_s) \\
&\quad + \int_s^t \mathbf{T}_{\epsilon,-1}(t-p)\big(\big(0, -(\theta/\epsilon)\check{v}(p)\big) + \mathbf{C}_{\epsilon,-1}(p)\big(\check{v}(p), \check{w}(p)\big) + \big(0, h(p)\big)\big)(dp \mid Z_{-1}).
\end{aligned}$$

By the uniqueness of the solution of (4.2), we obtain that $(\check{v}(\cdot), \check{w}(\cdot)) = (\tilde{v}(\cdot), \tilde{w}(\cdot))$. Finally, if $(v_s, w_s) \in Z_{-1}$, the conclusion follows from a density argument. $\square$

Now (4.1) and Lemma 4.1 with $h(t) = (\theta/\epsilon)\bar{v}(t)$, $t \in \mathbb{R}$, imply that

$$\big(\bar{v}(t), \bar{w}(t)\big) = \mathbf{U}_{\epsilon,-1}[\theta](t,s)\big(\bar{v}(s), \bar{w}(s)\big) + \int_s^t \mathbf{U}_{\epsilon,-1}[\theta](t,p)\big(0, (\theta/\epsilon)\bar{v}(p)\big)(dp \mid Z_{-1}). \quad (4.4)$$

Fix $\theta \geqslant \bar{\theta}_0$, where $\bar{\theta}_0$ is given by Proposition 3.7. Thanks to the decay estimate (3.15), we can let s tend to $-\infty$ in (4.4), so as to obtain

$$\big(\bar{v}(t), \bar{w}(t)\big) = \int_{-\infty}^t \mathbf{U}_{\epsilon,-1}[\theta](t,p)\big(0, (\theta/\epsilon)\bar{v}(p)\big)(dp \mid Z_{-1}) \quad (4.5)$$

for all $t \in \mathbb{R}$. Now observe that the mapping $p \mapsto (0, (\theta/\epsilon)\bar{v}(p))$ is continuous into Z_0. Therefore, thanks to the decay estimate (3.16), we deduce that

$$\big(\bar{v}(t), \bar{w}(t)\big) = \int_{-\infty}^{t} \mathbf{U}_{\epsilon,0}[\theta](t,p)\big(0, (\theta/\epsilon)\bar{v}(p)\big)(dp \mid Z_0). \tag{4.6}$$

It follows that $(\bar{v}(\cdot), \bar{w}(\cdot))$ is continuous into Z_0 and, for all $t \in \mathbb{R}$,

$$\begin{aligned}\big\|\big(\bar{v}(t), \bar{w}(t)\big)\big\|_{Z_{\epsilon,0}[\theta]} &\leqslant \int_{-\infty}^{t} Me^{-\delta(t-p)}\big\|\big(0, (\theta/\epsilon)\bar{v}(p)\big)\big\|_{Z_{\epsilon,0}[\theta]}\,dp \\ &\leqslant \int_{-\infty}^{t} Me^{-\delta(t-p)}\epsilon^{1/2}\big\|(\theta/\epsilon)\bar{v}(p)\big\|_{L^2}\,dp \leqslant \int_{-\infty}^{t} Me^{-\delta(t-p)}(\theta/\epsilon)R\,dp = \frac{MR\theta}{\delta\epsilon}.\end{aligned}$$

It follows that $(\bar{u}(\cdot), \bar{v}(\cdot))$ is continuously differentiable into Z_0, with

$$\begin{cases} (\partial_t \mid H_1)\bar{u}(t) = (\partial_t \mid H_0)\bar{u}(t) = \bar{v}(t), \\ \epsilon(\partial_t \mid H_0)\bar{v}(t) = \epsilon\bar{w}(t) = \epsilon(\partial_t \mid H_{-1})\bar{v}(t) = -\bar{v}(t) - \mathbf{A}_{-1}\bar{u}(t) + \hat{f}\big(\bar{u}(t)\big). \end{cases}$$

Now we have that

$$\mathbf{A}_{-1}\bar{u}(t) = -\epsilon\bar{w}(t) - \bar{v}(t) + \hat{f}\big(\bar{u}(t)\big). \tag{4.7}$$

The right-hand side of (4.7) is a continuous function of t into $L^2(\Omega)$. Then $\bar{u}(\cdot)$ is a continuous function into $D(\mathbf{A}_0)$, and

$$\big\|\mathbf{A}_0\bar{u}(t)\big\|_{L^2} \leqslant \epsilon\big\|\bar{w}(t)\big\|_{L^2} + \big\|\bar{v}(t)\big\|_{L^2} + \big\|\hat{f}\big(\bar{u}(t)\big)\big\|_{L^2}.$$

Summing up, we obtain that

$$\sup_{t\in\mathbb{R}}\big(\big\|\mathbf{A}_0\bar{u}(t)\big\|_{L^2}^2 + \big\|\bar{v}(t)\big\|_{H_0^1}^2 + \epsilon\big\|(\partial_t \mid H_0)\bar{v}(t)\big\|_{L^2}^2\big) \leqslant 4\frac{MR\theta}{\delta\epsilon} + \tilde{C}^2\big(1+R^3\big)^2.$$

This concludes the proof of Theorem 2.8.

5. Proof of Theorem 2.12

Let R be a positive constant. Throughout this section, for every $\epsilon \in\,]0,1]$, we denote by $(\bar{u}_\epsilon(\cdot), \bar{v}_\epsilon(\cdot)) : \mathbb{R} \to Z_0$ a fixed bounded full solution of (2.11), such that $\sup_{t\in\mathbb{R}}(\|\bar{u}_\epsilon(t)\|_{H_0^1}^2 + \epsilon\|\bar{v}_\epsilon(t)\|_{L^2}^2) \leqslant R$. We need the following lemma.

Lemma 5.1. *There exists a positive constant K such that, for all $\epsilon \in\,]0,1]$,*

$$\int_{-\infty}^{+\infty}\big\|\bar{v}_\epsilon(t)\big\|_{L^2}^2\,dt \leqslant K.$$

The constant K depends only on R and on the constants in Hypotheses 2.1 *and* 2.11. *In particular, K is independent of ϵ.*

Proof. Define the standard Lyapunov functional

$$L(u,v) := \epsilon\frac{1}{2}\|v\|^2_{L^2} + \frac{1}{2}\|u\|^2_{H^1_0} - \int_\Omega F\big(x,u(x)\big)\,dx, \quad (u,v)\in H^1_0(\Omega)\times L^2(\Omega),$$

where $F(x,u) := \int_0^u f(x,s)\,ds$. Then the mapping $t\mapsto L(\bar u(t),\bar v(t))$ is differentiable and

$$\frac{d}{dt}L\big(\bar u(t),\bar v(t)\big) = -\big\|\bar v(t)\big\|^2_{L^2}$$

(for details, see the proof of Proposition 4.1 in [15]). Then, for every $t_1<t_2$,

$$\int_{t_1}^{t_2}\big\|\bar v_\epsilon(t)\big\|^2_{L^2}\,dt \leqslant \big|L\big(\bar u(t_1),\bar v(t_1)\big)\big| + \big|L\big(\bar u(t_1),\bar v(t_2)\big)\big| \leqslant K(R),$$

where $K(R)$ is a suitable constant depending on R and on the constants of Hypothesis 2.11. □

It follows from Theorem 2.8 that $(\bar u_\epsilon(\cdot),\bar v_\epsilon(\cdot))$ is continuous into Z_1 and continuously differentiable into Z_0, with

$$\begin{cases} (\partial_t \mid H_1)\bar u_\epsilon(t) = \bar v_\epsilon(t), \\ \epsilon(\partial_t \mid H_0)\bar v_\epsilon(t) = -\bar v_\epsilon(t) - \mathbf{A}_0\bar u_\epsilon(t) + \hat f\big(\bar u_\epsilon(t)\big). \end{cases} \tag{5.1}$$

Moreover, for every $\epsilon\in\,]0,1]$ there exists a positive constant $\tilde R_\epsilon$ such that

$$\sup_{t\in\mathbb{R}}\big(\big\|\mathbf{A}_0\bar u_\epsilon(t)\big\|^2_{L^2} + \big\|\bar v_\epsilon(t)\big\|^2_{H^1_0} + \epsilon\big\|(\partial_t \mid H_0)\bar v_\epsilon(t)\big\|^2_{L^2}\big) \leqslant \tilde R_\epsilon. \tag{5.2}$$

Set $\bar w(t) := (\partial_t \mid H_0)\bar v(t)$, $t\in\mathbb{R}$. Using (2.3) we see that $(\bar v_\epsilon(\cdot),\bar w_\epsilon(\cdot))$ is continuously differentiable into Z_{-1}, and

$$\begin{cases} (\partial_t \mid H_0)\bar v_\epsilon(t) = \bar w_\epsilon(t), \\ \epsilon(\partial_t \mid H_{-1})\bar w_\epsilon(t) = -\bar w_\epsilon(t) - \mathbf{A}_{-1}\bar v_\epsilon(t) + \widehat{\partial_u f}\big(\bar u_\epsilon(t)\big)\cdot\bar v_\epsilon(t). \end{cases}$$

Let $\theta\geqslant 0$. Since the mapping $t\mapsto (0,\widehat{\partial_u f}(\bar u_\epsilon(t))\cdot\bar v_\epsilon(t))$ is continuous into $Z_0 = D(\mathbf{B}_{\epsilon,-1}[\theta])$, it follows from Theorem 2.6 that, for $s,t\in\mathbb{R}$, with $s\leqslant t$, $(\bar v_\epsilon(\cdot),\bar w_\epsilon(\cdot))$ satisfies the equality

$$\begin{aligned}\big(\bar v_\epsilon(t),\bar w_\epsilon(t)\big) &= \mathbf{T}_{\epsilon,-1}[\theta](t-s)\big(\bar v_\epsilon(s),\bar w_\epsilon(s)\big) \\ &\quad + \int_s^t \mathbf{T}_{\epsilon,-1}[\theta](t-p)\big(0,(1/\epsilon)\widehat{\partial_u f}\big(\bar u_\epsilon(p)\big)\cdot\bar v_\epsilon(p) + (\theta/\epsilon)\bar v_\epsilon(p)\big)(dp \mid Z_{-1}).\end{aligned}$$

Finally, since $(\bar v_\epsilon(\cdot),\bar w_\epsilon(\cdot))$ is continuous into Z_0, it follows that $(\bar v_\epsilon(\cdot),\bar w_\epsilon(\cdot))$ satisfies the equality

$$\begin{aligned}\big(\bar v_\epsilon(t), \bar w_\epsilon(t)\big) &= \mathbf{T}_{\epsilon,0}[\theta](t-s)\big(\bar v_\epsilon(s), \bar w_\epsilon(s)\big)\\ &\quad + \int_s^t \mathbf{T}_{\epsilon,0}[\theta](t-p)\big(0, (1/\epsilon)\widehat{\partial_u f}\big(\bar u_\epsilon(p)\big)\cdot \bar v_\epsilon(p) + (\theta/\epsilon)\bar v_\epsilon(p)\big)(dp \mid Z_0). \quad (5.3)\end{aligned}$$

Let δ be a positive constant, with $\delta \leqslant \min\{1/2, \lambda_1/2\}$. We define the following energy functional on Z_0:

$$\tilde E_{\epsilon,\theta}(v,w) := \frac{1}{2}\epsilon\|\delta v + w\|^2_{L^2} + \frac{1}{2}\|v\|^2_{H^1_0[\theta]} + \frac{1}{2}\big(\epsilon\delta^2 - \delta\big)\|v\|^2_{L^2}. \quad (5.4)$$

A direct computation using (3.5) shows that, for all $\theta \geqslant 0$,

$$\frac{1}{4}\big\|(v,w)\big\|_{Z_{\epsilon,0}[\theta]} \leqslant \tilde E_{\epsilon,\theta}(v,w) \leqslant \frac{3}{4}\big\|(v,w)\big\|_{Z_{\epsilon,0}[\theta]}, \quad (v,w) \in Z_0. \quad (5.5)$$

Moreover, by Lemma 3.5, for every ρ, with $0<\rho<1$, there exists $\theta_\rho > 0$ such that, for all $\theta \geqslant \theta_\rho$, all $t \in \mathbb{R}$ and all $(v,w) \in Z_0$,

$$(1-\rho)\tilde E_{\epsilon,\theta}(v,w) \leqslant \tilde E_{\epsilon,\theta}(v,w) + \frac{1}{2}\int_\Omega \widehat{\partial_u f}\big(\bar u_\epsilon(t)\big)(x)\big|v(x)\big|^2\,dx \leqslant (1+\rho)\tilde E_{\epsilon,\theta}(v,w). \quad (5.6)$$

Fixing $\rho = 1/2$ and setting $\theta_* := \theta_{1/2}$, we obtain from (5.5) and (5.6) that, for all $\theta \geqslant \theta_*$, all $t\in\mathbb{R}$ and all $(v,w)\in Z_0$,

$$\frac{1}{8}\big\|(v,w)\big\|_{Z_{\epsilon,0}[\theta]} \leqslant \tilde E_{\epsilon,\theta}(v,w) + \frac{1}{2}\int_\Omega \widehat{\partial_u f}\big(\bar u_\epsilon(t)\big)(x)\big|v(x)\big|^2\,dx \leqslant \frac{9}{8}\big\|(v,w)\big\|_{Z_{\epsilon,0}[\theta]}. \quad (5.7)$$

We define the following function:

$$\Lambda_{\epsilon,\theta}(t) := E_{\epsilon,\theta}\big(\bar v_\epsilon(t), \bar w_\epsilon(t)\big) + \frac{1}{2}\int_\Omega \widehat{\partial_u f}\big(\bar u_\epsilon(t)\big)(x)\big|\bar v_\epsilon(t)(x)\big|^2\,dx. \quad (5.8)$$

We need the following lemma, whose proof is left to the reader:

Lemma 5.2. *Assume Hypothesis* 2.11. *Define the mapping*

$$\mathcal{G}_\epsilon(t) := \frac{1}{2}\int_\Omega \widehat{\partial_u f}\big(\bar u_\epsilon(t)\big)(x)\big|\bar v_\epsilon(t)(x)\big|^2\,dx.$$

Then $\mathcal{G}_\epsilon(\cdot)$ is continuously differentiable, and

$$\begin{aligned}\frac{d}{dt}\mathcal{G}_\epsilon(t) &= \frac{1}{2}\int_\Omega \widehat{\partial_{uu} f}\big(\bar u_\epsilon(t)\big)(x)\bar v_\epsilon(t)(x)\big|\bar v_\epsilon(t)(x)\big|^2\,dx\\ &\quad + \int_\Omega \widehat{\partial_u f}\big(\bar u_\epsilon(t)\big)(x)\bar v_\epsilon(t)(x)\bar w_\epsilon(t)(x)\,dx.\end{aligned}$$

Using (5.3), Theorem 2.6 in [15] and Lemma 5.2, we see that $\Lambda_{\epsilon,\theta}(\cdot)$ is differentiable and

$$\begin{aligned}
&\frac{d}{dt}\Lambda_{\epsilon,\theta}(t)+2\delta\Lambda_{\epsilon,\theta}(t)\\
&\quad=(2\delta\epsilon-1)\left\|\bar{w}_\epsilon(t)+\delta\bar{v}_\epsilon(t)\right\|_{L^2}^2\\
&\qquad+\left\langle\bar{w}_\epsilon(t)+\delta\bar{v}_\epsilon(t),\theta\bar{v}_\epsilon(t)\right\rangle_{L^2}+\frac{1}{2}\int_\Omega\widehat{\partial_{uu}f}\big(\bar{u}_\epsilon(t)\big)(x)\bar{v}_\epsilon(t)(x)\left|\bar{v}_\epsilon(t)(x)\right|^2dx. \qquad (5.9)
\end{aligned}$$

By Hypothesis 2.11, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega\widehat{\partial_{uu}f}\big(\bar{u}_\epsilon(t)\big)(x)\bar{v}_\epsilon(t)(x)\left|\bar{v}_\epsilon(t)(x)\right|^2dx\\
&\quad\leqslant\frac{1}{2}\int_\Omega C_1\big(1+\left|\bar{u}_\epsilon(t)(x)\right|\big)\left|\bar{v}_\epsilon(t)(x)\right|\left|\bar{v}_\epsilon(t)(x)\right|^2dx\\
&\quad\leqslant\frac{1}{2}C_1\left\|\bar{v}_\epsilon(t)\right\|_{L^2}\left\|\bar{v}_\epsilon(t)\right\|_{L^4}^2+\frac{1}{2}C_1\left\|\bar{u}_\epsilon(t)\right\|_{L^6}\left\|\bar{v}_\epsilon(t)\right\|_{L^2}\left\|\bar{v}_\epsilon(t)\right\|_{L^6}^2\\
&\quad\leqslant\frac{1}{2}C_2(1+R)\left\|\bar{v}_\epsilon(t)\right\|_{L^2}\left\|\bar{v}_\epsilon(t)\right\|_{H_0^1}^2.
\end{aligned}$$

It follows that, for every $\nu>0$, there exists $C_\nu>0$ such that

$$\begin{aligned}
\frac{1}{2}\int_\Omega\widehat{\partial_{uu}f}\big(\bar{u}_\epsilon(t)\big)(x)\bar{v}_\epsilon(t)(x)\left|\bar{v}_\epsilon(t)(x)\right|^2dx&\leqslant\nu\left\|\bar{v}_\epsilon(t)\right\|_{H_0^1}^2+C_\nu\left\|\bar{v}_\epsilon(t)\right\|_{L^2}^2\left\|\bar{v}_\epsilon(t)\right\|_{H_0^1}^2\\
&\leqslant\nu\left\|\bar{v}_\epsilon(t)\right\|_{H_0^1[\theta]}^2+C_\nu\left\|\bar{v}_\epsilon(t)\right\|_{L^2}^2\left\|\bar{v}_\epsilon(t)\right\|_{H_0^1[\theta]}^2.
\end{aligned}$$

Then, choosing $\nu\leqslant\delta$ and using Cauchy–Schwartz inequality in (5.9), we get

$$\frac{d}{dt}\Lambda_{\epsilon,\theta}(t)+\delta\Lambda_{\epsilon,\theta}(t)\leqslant\big(\theta^2/2\big)\left\|\bar{v}_\epsilon(t)\right\|_{L^2}^2+2C_\nu\left\|\bar{v}_\epsilon(t)\right\|_{L^2}^2\Lambda_{\epsilon,\theta}(t). \qquad (5.10)$$

Let $\sigma,t,\tau\in\mathbb{R}$, with $\sigma\geqslant t\geqslant\tau$. We multiply (5.10) by $e^{\delta(t-\tau)-\int_\tau^t 2C_\nu\|\bar{v}_\epsilon(s)\|_{L^2}^2\,ds}$ and we obtain that

$$\frac{d}{dt}\big(e^{\delta(t-\tau)-\int_\tau^t 2C_\nu\|\bar{v}_\epsilon(s)\|_{L^2}^2\,ds}\Lambda_{\epsilon,\theta}(t)\big)\leqslant\big(\theta^2/2\big)e^{\delta(t-\tau)-\int_\tau^t 2C_\nu\|\bar{v}_\epsilon(s)\|_{L^2}^2\,ds}\left\|\bar{v}_\epsilon(t)\right\|_{L^2}^2.$$

Integrating on $[\tau,\sigma]$, we get

$$\begin{aligned}
&e^{\delta(\sigma-\tau)-\int_\tau^\sigma 2C_\nu\|\bar{v}_\epsilon(s)\|_{L^2}^2\,ds}\Lambda_{\epsilon,\theta}(\sigma)\\
&\quad\leqslant\Lambda_{\epsilon,\theta}(\tau)+\big(\theta^2/2\big)\int_\tau^\sigma e^{\delta(t-\tau)-\int_\tau^t 2C_\nu\|\bar{v}_\epsilon(s)\|_{L^2}^2\,ds}\left\|\bar{v}_\epsilon(t)\right\|_{L^2}^2\,dt.
\end{aligned}$$

It follows from Lemma 5.1 that

$$\Lambda_{\epsilon,\theta}(\sigma) \leqslant e^{-\delta(\sigma-\tau)+\int_\tau^\sigma 2C_\nu \|\bar{v}_\epsilon(s)\|_{L^2}^2\,ds}\Lambda_{\epsilon,\theta}(\tau)$$
$$+\left(\theta^2/2\right)\int_\tau^\sigma e^{-\delta(\sigma-t)+\int_t^\sigma 2C_\nu\|\bar{v}_\epsilon(s)\|_{L^2}^2\,ds}\left\|\bar{v}_\epsilon(t)\right\|_{L^2}^2\,dt$$
$$\leqslant e^{-\delta(\sigma-\tau)}e^{2C_\nu K}\Lambda_{\epsilon,\theta}(\tau)+\left(\theta^2/2\right)Ke^{2C_\nu K}.$$

Using (5.7), (5.2) and (3.3) we get

$$(1/8)\left\|\left(\bar{v}_\epsilon(\sigma),\bar{w}_\epsilon(\sigma)\right)\right\|_{Z_{\epsilon,0}[\theta]}^2$$
$$\leqslant (9/8)e^{-\delta(\sigma-\tau)}e^{2C_\nu K}\left\|\left(\bar{v}_\epsilon(\tau),\bar{w}_\epsilon(\tau)\right)\right\|_{Z_{\epsilon,0}[\theta]}^2+\left(\theta^2/2\right)Ke^{2C_\nu K}$$
$$\leqslant (9/8)\left(\frac{\theta+\lambda_1}{\lambda_1}\right)e^{-\delta(\sigma-\tau)}e^{2C_\nu K}\tilde{R}_\epsilon+\left(\theta^2/2\right)Ke^{2C_\nu K}.$$

Letting τ tend to $-\infty$, we finally get

$$\left\|\left(\bar{v}_\epsilon(\sigma),\bar{w}_\epsilon(\sigma)\right)\right\|_{Z_{\epsilon,0}}^2 \leqslant 4\theta^2Ke^{2C_\nu K},\quad \sigma\in\mathbb{R}. \tag{5.11}$$

This last inequality, together with (5.1), yields

$$\sup_{t\in\mathbb{R}}\left(\left\|\mathbf{A}_0\bar{u}_\epsilon(t)\right\|_{L^2}^2+\left\|\bar{v}_\epsilon(t)\right\|_{H_0^1}^2+\epsilon\left\|(\partial_t \mid H_0)\bar{v}_\epsilon(t)\right\|_{L^2}^2\right)\leqslant\tilde{R},$$

where $\tilde{R}$ depends only on the constants in Hypotheses 2.1 and 2.11 and on R. This concludes the proof of Theorem 2.12.

6. An application: Upper semicontinuity of attractors

In this section we assume Hypotheses 2.1 and 2.11. Moreover, we make the following structure assumption on the nonlinearity $f(x,u)$.

Hypothesis 6.1. There exist a positive number μ and a function $c(\cdot)\in L^2(\Omega)$ such that:

(1) $f(x,u)u-\mu F(x,u)\leqslant c(x)$;
(2) $F(x,u)\leqslant c(x)$.

Here, $F(x,u):=\int_0^u f(x,s)\,ds$, $(x,u)\in\Omega\times\mathbb{R}$.

It was proved in [15] that under Hypotheses 2.1, 2.11 and 6.1, for every $\epsilon\in\,]0,1]$, Eq. (2.8) (more precisely: its mild formulation (2.11)) generates a global semiflow in $H_0^1(\Omega)\times L^2(\Omega)$, possessing a compact global attractor $\mathcal{A}_\epsilon$. Moreover, there exists a positive constant R such that

$$\sup_{\epsilon\in]0,1]}\sup\left\{\|u\|_{H_0^1}^2+\epsilon\|v\|_{L^2}^2 \;\middle|\; (u,v)\in\mathcal{A}_\epsilon\right\}\leqslant R.$$

Consider now the formal limit of (2.8) as $\epsilon\to 0$, i.e. the parabolic equation

$$u_t + \beta(x)u - \sum_{ij} \bigl(a_{ij}(x)u_{x_j}\bigr)_{x_i} = f(x,u), \quad (t,x) \in [0,+\infty[\times \Omega,$$
$$u = 0, \quad (t,x) \in [0,+\infty[\times \partial\Omega, \tag{6.1}$$

with Cauchy datum $u(0) = u_0$. Again we rewrite (6.1) as an integral evolution equation in the space $H_0^1(\Omega)$, namely

$$u(t) = e^{-\mathbf{A}_0 t} u_0 + \int_0^t e^{-\mathbf{A}_0 (t-s)} \hat{f}\bigl(u(s)\bigr)\, ds, \tag{6.2}$$

where $e^{-\mathbf{A}_0 t}$, $t \geqslant 0$, is the analytic semigroup generated by the positive selfadjoint operator $\mathbf{A}_0$ in $L^2(\Omega)$. It was proved in [14] that under Hypotheses 2.1, 2.11 and 6.1, Eq. (6.1) (more precisely: its mild formulation (6.2)) generates a global semiflow in $H_0^1(\Omega)$, possessing a compact global attractor $\tilde{\mathcal{A}}$. Moreover, $\tilde{\mathcal{A}} \subset D(\mathbf{A}_0)$ and $\tilde{\mathcal{A}}$ is compact in $D(\mathbf{A}_0)$.

Let $\Gamma : D(\mathbf{A}_0) \to H_0^1(\Omega) \times L^2(\Omega)$ be defined by $\Gamma(u) := (u, \mathbf{A}_0 u + \hat{f}(u))$. Set $\mathcal{A}_0 := \Gamma(\tilde{\mathcal{A}})$. In [16] the following result was proved.

Theorem 6.2. *(See Theorem 1.4 in [16].) The family $(\mathcal{A}_\epsilon)_{\epsilon \in [0,1]}$ is upper semicontinuous at $\epsilon = 0$ with respect to the topology of $H_0^1(\Omega) \times H^{-1}(\Omega)$, i.e.*

$$\lim_{\epsilon \to 0} \sup_{y \in \mathcal{A}_\epsilon} \inf_{z \in \mathcal{A}_0} \|y - z\|_{H_0^1 \times H^{-1}} = 0.$$

This result is not completely satisfactory. The optimal result would be to obtain upper semicontinuity with respect to the topology of $H_0^1(\Omega) \times L^2(\Omega)$. Actually, thanks to Theorem 2.12, we are now able to prove the optimal result.

Theorem 6.3. *The family $(\mathcal{A}_\epsilon)_{\epsilon \in [0,1]}$ is upper semicontinuous at $\epsilon = 0$ with respect to the topology of $H_0^1(\Omega) \times L^2(\Omega)$, i.e.*

$$\lim_{\epsilon \to 0} \sup_{y \in \mathcal{A}_\epsilon} \inf_{z \in \mathcal{A}_0} \|y - z\|_{H_0^1 \times L^2} = 0.$$

Indeed, the main ingredient in the proof of Theorem 4.1 in [16] is the following theorem.

Theorem 6.4. *(See Theorem 3.8 in [16].) Let $(\epsilon_n)_n$ be a sequence of positive numbers converging to 0. For each $n \in \mathbb{N}$ let $z_n = (u_n, v_n) : \mathbb{R} \to H_0^1(\Omega) \times L^2(\Omega)$ be a bounded full solution of (2.11) such that*

$$\sup_{n \in \mathbb{N}} \sup_{t \in \mathbb{R}} \bigl(\|u_n(t)\|_{H_0^1}^2 + \epsilon_n \|v_n(t)\|_{L^2}^2 \bigr) \leqslant R < \infty.$$

Then a subsequence of $(z_n)_n$ converges in $H_0^1(\Omega) \times H^{-1}(\Omega)$, uniformly on compact subsets of $\mathbb{R}$, to a function $z : \mathbb{R} \to H_0^1(\Omega) \times L^2(\Omega)$ with $z = (u, v)$, where u is a solution of (6.2) and $v = (\partial_t \mid L^2(\Omega))u$.

If in Theorem 6.4 we assume also that, for each $n \in \mathbb{N}$, the function $u_n(\cdot)$ is uniformly continuous, then it follows from Theorem 2.12 that the sequence $(v_n(\cdot))_n$ is bounded in $L^\infty(\mathbb{R}, H_0^1(\Omega))$. Interpolation between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ then implies that $v_n(t) \to v(t)$ in $L^2(\Omega)$ uniformly for t lying in compact subsets of $\mathbb{R}$. Now using Lemma 2.9 and an obvious contradiction argument one easily completes the proof of Theorem 6.3.

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